A CONVERGENCE THEOREM FOR CONTINUED FRACTIONS*

BY

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1. **Introduction.** This paper is concerned with the question of convergence of continued fractions of the form

$$(1.1) \qquad \frac{1}{1\,+}\ \frac{a_2}{1\,+}\ \frac{a_3}{1\,+}\ \frac{a_4}{1\,+}\ \cdots$$

in which the a_n are complex numbers. Our investigation began with a comparison of the infinite series

$$(1.2) \qquad \frac{A_1}{B_1} + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \left(\frac{A_3}{B_3} - \frac{A_2}{B_2}\right) + \cdots,$$

equivalent to the sequence of approximants of the continued fraction, with a convergent positive term series $\sum p_n$. It was found that if $r_n = p_{n+1}/p_n$, and

$$(1.3) \qquad r_n\,|\,1 + a_n + a_{n+1}\,| \geqq r_n r_{n-2}\,|\,a_n\,| + |\,a_{n+1}\,|,$$

$n = 1, 2, 3, \cdots, r_0 = r_{-1} = a_1 = 0$, then the series $\sum p_n$ is a majorant for (1.2), so that the continued fraction converges.

Later, in proving that the Pringsheim convergence theorem can be obtained from the above by specializing the series $\sum p_n$, it was discovered that the inequalities (1.3) have a second interpretation. In fact, if $a_n \neq 0$, and

$$(1.4) \qquad y_0 + \frac{x_1}{y_1\,+}\ \frac{x_2}{y_2\,+}\ \frac{x_3}{y_3\,+}\ \cdots$$

denotes the even (odd) "contraction" of (1.1), then the inequalities (1.3) for odd (even) values of n may be written in the form

$$(1.5) \qquad |\,c_1 y_1\,| \geqq 1, \qquad |\,c_n y_n\,| \geqq |\,c_{n-1} c_n y_n\,| + 1,$$

$n = 2, 3, 4, \cdots$, where $c_n \neq 0$. But these are precisely the *Pringsheim relations*[†] for the continued fraction (1.4). Hence if the inequalities (1.3) hold for some $r_n > 0$ (not necessarily related to a convergent series $\sum p_n$), the

sequences of even and odd approximants of (1.1) have limits, finite or infinite, and the limit of the even (odd) approximants is finite if inequality holds in (1.3) for an odd (even) index. *Thus the relations* (1.3) *have two interpretations, and therefore are extremely important instruments in the convergence theory.*

One of the most important results obtained in the paper is the "parabola theorem," namely, that if the elements a_n of (1.1) lie within or upon the parabola $|z| - \Re(z) = 1/2$, then the continued fraction converges if and only if some a_n is 0, or the series $\sum |b_n|$ diverges, where $b_1 = 1$, $a_n = 1/b_{n-1}b_n$, $n = 2, 3, 4, \cdots$. In the first announcement of the theorem,* the a_n were restricted to lie in a closed bounded region entirely within the parabola, a result which was obtained using $r_n = r < 1$ in (1.3). Using the *second* interpretation of the inequalities, we were able later to remove the restriction requiring the a_n to lie in the interior of the parabola, and to replace the boundedness condition by the condition on the series $\sum |b_n|$.

The parabola cannot be replaced by a "better" curve symmetric with respect to the real axis. *We thus characterize completely all regions in the complex plane, symmetric with respect to the real axis, in which the elements a_n may vary independently while the continued fraction remains convergent.* We have therefore, in a certain sense, extended as far as possible the theorem given by Worpitzky† three-quarters of a century ago, namely, that (1.1) converges if $|a_n| \leq 1/4$, $n = 2, 3, 4, \cdots$.

Other results obtained include: a notable improvement in the Pringsheim theorem on uniform convergence (Theorem B), the "triangle theorem" (Theorem G), some results on the *problem of Szász* (§6), and the theorem that if (1.3) holds and the a_n are bounded, then (1.1) converges (Theorem E).

The methods used throughout the paper are elementary in character, and are based strictly upon the continued fraction algorithm.

2. **The first interpretation of the fundamental inequalities.** The basic result of this paper is embodied in the theorem which follows.

THEOREM A. *Let* (1.1) *be a continued fraction with complex elements a_n, and nth approximant A_n/B_n. Suppose that for some nonnegative numbers r_n the inequalities* (1.3) *hold. Then $B_n \neq 0$, $n = 1, 2, 3, \cdots$, and the series $1 + \sum r_1 r_2 \cdots r_n$ is a majorant for the series* (1.2). *The continued fraction converges if some a_n is 0, or if the series $1 + \sum r_1 r_2 \cdots r_n$ converges. The sum of the latter series is an upper bound for the absolute value of the continued fraction.*

Proof. By (1.3) we have $B_2 = 1 + a_2 \neq 0$, $B_3 = 1 + a_2 + a_3 \neq 0$; and if we

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* Bulletin of the American Mathematical Society, abstract 45-7-277.

† Worpitzky, Jahresbericht, Friedrichs-Gymnasium und Realschule, Berlin, 1865, pp. 3–39. Independent proofs of this theorem were given later by Pringsheim and by Van Vleck.

put $c_n = a_{n+1}B_{n-1}/B_{n+1}$, $n = 1, 2, 3, \cdots$, we have $|c_1| = |a_2/(1+a_2)| \leq r_1$, $|c_2| = |a_3/(1+a_2+a_3)| \leq r_2$. Let us assume that $B_{n+1} \neq 0$, $|c_n| \leq r_n$, for $n = 1, 2, 3, \cdots, k$, $k \geq 2$, and prove that the same holds for $n = k+1$, and hence for all n. Inasmuch as

$$\left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| = |c_1 c_2 \cdots c_n|,$$

it will follow that $1 + \sum r_1 r_2 \cdots r_n$ is a majorant for the series (1.2), so that the continued fraction converges if $1 + \sum r_1 r_2 \cdots r_n$ converges. Since $B_n \neq 0$, for all n, the continued fraction converges if some a_n is 0.

From the recursion relation $B_n = B_{n-1} + a_n B_{n-2}$, $n \geq 2$, we easily obtain the formula

(2.1) $$B_{k+2} = (1 + a_{k+1} + a_{k+2})B_k - a_k a_{k+1} B_{k-2},$$

$k = 2, 3, 4, \cdots$. If $a_{k+2} \neq 0$, so that $r_{k+1} > 0$, then we have

$$\frac{B_{k+2}}{a_{k+2}B_k} = \frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}} - \frac{a_{k+1}}{a_{k+2}} \frac{a_k B_{k-2}}{B_k},$$

or, by (1.3) and our assumption,

$$\left| \frac{B_{k+2}}{a_{k+2}B_k} \right| \geq \left| \frac{1 + a_{k+1} + a_{k+2}}{a_{k+2}} \right| - \left| \frac{a_{k+1}}{a_{k+2}} \right| r_{k-1} \geq \frac{1}{r_{k+1}} > 0.$$

Hence $B_{k+2} \neq 0$, and $|c_{k+1}| \leq r_{k+1}$. On the other hand, if $a_{k+2} = 0$, it follows from the fundamental recursion formula that $B_{k+2} = B_{k+1} \neq 0$, and $|c_{k+1}| = 0 \leq r_{k+1}$. This completes the induction and the proof of Theorem A.

Since Theorem A is simply a comparison theorem for the series (1.2), we have at once by the *Weierstrass "M-test,"* the following theorem on *uniform convergence* of continued fractions.

Theorem 2.1. *If in the continued fraction* (1.1) *of Theorem A, the elements a_n are functions of any variables over a domain D in which the inequalities* (1.3) *hold, with the r_n independent of the variables, and if the series $1 + \sum r_1 r_2 \cdots r_n$ converges, then the continued fraction converges uniformly over D.*

If in Theorem A we use the series

(2.2) $$\frac{A_{k+1}}{B_{k+1}} + \left(\frac{A_{k+2}}{B_{k+2}} - \frac{A_{k+1}}{B_{k+1}} \right) + \left(\frac{A_{k+3}}{B_{k+3}} - \frac{A_{k+2}}{B_{k+2}} \right) + \cdots$$

instead of the series (1.2) we get by the same methods the more general result:

THEOREM A'. *If there exist an index $k \geq 0$, and numbers $r_n \geq 0$, $n = k+1$, $k+2$, $k+3$, $\cdots$, such that A_{k+1}, B_{k+2}, $B_{k+3} \neq 0$ and*

$$r_{k+1} | B_{k+2} A_{k+1} | \geq | a_2 a_3 \cdots a_{k+2} |, \qquad r_{k+2} | B_{k+3} | \geq | a_{k+3} B_{k+1} |,$$

$$r_n | 1 + a_n + a_{n+1} | \geq r_n r_{n-2} | a_n | + | a_{n+2} |,$$

$n = k+3$, $k+4$, $k+5$, $\cdots$, then the series

$$(2.3) \qquad \left| \frac{A_{k+1}}{B_{k+1}} \right| \left(1 + \sum_{n=k+1}^{\infty} r_{k+1} r_{k+2} \cdots r_n \right)$$

is a majorant for the series (2.2). *The continued fraction converges if some a_n is 0 for $n \geq k+2$, or if the series* (2.3) *converges.*

3. **Relation of Theorem A to other convergence criteria.** We now prove a second theorem.

THEOREM B. *Let p_1, p_2, p_3, $\cdots$ be real and positive, $p_1 > 1$. Then the continued fraction* (1.1) *in which the elements a_n are functions of any variables converges uniformly over the domain characterized by the inequalities*

$$(3.1) \qquad | a_n | \leq (p_n - 1)/p_n p_{n-1}, \qquad n = 2, 3, 4, \cdots.$$

The quantity

$$(3.2) \qquad 1 + \frac{p_1}{p_1 - 1} \left\{ \frac{1}{p_1} - \frac{1}{1 + \sum_1^\infty (p_1 - 1)(p_2 - 1) \cdots (p_n - 1)} \right\}$$

is an upper bound for the absolute value of the continued fraction over the domain, and is actually attained if $a_n = (1 - p_n)/p_n p_{n-1}$, $n = 2, 3, 4, \cdots$.

This theorem includes the *general Pringsheim criterion*.[†] However, our theorem goes farther and gives an upper bound for the absolute value of the continued fraction. The greatest improvement is in respect to the question of uniform convergence. *Pringsheim required that the series* $\sum (p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$ *be divergent in order to obtain uniform convergence of the continued fraction.*[‡]

Proof of Theorem B. Let $t_n = (p_n - 1)p_n p_{n-1}$, and form the continued fraction

$$(3.3) \qquad 1/1 - t_2/1 - t_3/1 - \cdots.$$

Let G_n/H_n be the nth convergent of (3.3) and put

$$H_n^* = p_1 p_2 \cdots p_n H_n, \quad n = 1, 2, 3, \cdots, \qquad H_0^* = 1,$$

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† Perron, p. 258, Theorem 26. It is to be noted that there is no loss in generality in assuming $p_1 > 1$ in that theorem.

‡ Perron, p. 262, Theorem 30.

so that $H_n^* = p_n H_{n-1}^* - p_n p_{n-1} t_n H_{n-2}^*$, $n = 2, 3, 4, \cdots$, and

$$H_n^* - H_{n-1}^* = (p_n - 1)(H_{n-1}^* - H_{n-2}^*) = (p_1 - 1)(p_2 - 1) \cdots (p_n - 1),$$

$n = 1, 2, 3, \cdots$. Thus $H_n^* \geqq 1$, and consequently $H_n > 0$ for all n.

We take for r_n in Theorem A the quantity

$$r_n = t_{n+1} H_{n-1} / H_{n+1}, \qquad n = 1, 2, 3, \cdots.$$

Inasmuch as $H_{n+1} = (1 - t_n - t_{n+1}) H_{n-1} - t_n t_{n-1} H_{n-3}$, we readily verify that $r_n(1 - t_n - t_{n+1}) = r_n r_{n-2} t_n + t_{n+1}$, $n = 3, 4, 5, \cdots$, $(r_0 = r_{-1} = t_1 = 0)$. Hence it follows from (3.1) and the definition of t_n that the elements of the continued fraction satisfy the inequalities of Theorem A with the values of the r_n which we have determined.

Since

$$r_1 r_2 \cdots r_n = \frac{p_1}{p_1 - 1}\left[\frac{1}{H_n^*} - \frac{1}{H_{n+1}^*}\right],$$

it follows that the series $1 + \sum r_1 r_2 \cdots r_n$ converges; and since the r_n are independent of the variables, it follows by Theorem 2.1 that the continued fraction converges uniformly over the domain of the variables.

The quantity (3.2) is simply the sum of the majorant series, and is therefore an upper bound for the absolute value of the continued fraction. It is attained by the continued fraction when $a_n = -t_n$.

This completes the proof of Theorem B.

By specializing the p_n in Theorem B the familiar *special Pringsheim criteria*[†] may be obtained, with the attendant upper bounds. For example, setting $p_n = 2$ we obtain this theorem:

THEOREM 3.1. (Worpitzky.) *The continued fraction* (1.1) *converges if* $|a_n| \leqq 1/4$, $n = 2, 3, 4, \cdots$.

In this case the upper bound for the absolute value of the continued fraction is 2, and if $a_n = -1/4$, this value is attained.

The importance of Theorem B is illustrated by the fact that the proof of the following theorem of Van Vleck as given by Perron[‡] can now be considerably shortened.

THEOREM 3.2. (Van Vleck.) *Let* $s_1, s_2, s_3, \cdots$ *be real numbers, and* $t_1, t_2, t_3, \cdots$ *complex numbers, such that*

$$0 < s_1 < 1, \qquad 0 \leqq s_n < 1, \qquad n = 2, 3, 4, \cdots,$$
$$|t_n| \leqq 1, \qquad n = 1, 2, 3, \cdots;$$

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† Perron, pp. 259–260.

‡ Perron, pp. 262–264.

and let x be a complex variable. Then the continued fraction

$$\frac{s_1 t_1 x^{n_1}}{1} + \frac{s_2(1 - s_1)t_2 x^{n_2}}{1} + \frac{s_3(1 - s_2)t_3 x^{n_3}}{1} + \cdots$$

(n_i positive integers) converges uniformly for $|x| \leqq 1$. The modulus of the analytic function which the continued fraction represents cannot exceed

$$s_1 \left| t_1 x^{n_2} \right| \left(1 + \frac{1 - s_1}{s_1} \left[1 - \frac{1}{1 + (1 - s_1) \sum_1^\infty \dfrac{s_1 s_2 \cdots s_n}{(1 - s_1)(1 - s_2) \cdots (1 - s_n)}} \right] \right).$$

Proof. Take $p_n = 1/(1 - s_n)$, $n = 1, 2, 3, \cdots$, in Theorem B.

As an illustration of the last theorem, the function $x/(\arctan x)$ has the continued fraction expansion*

$$1 + \frac{\dfrac{x^2}{1 \cdot 3}}{1} + \frac{\dfrac{4x^2}{3 \cdot 5}}{1} + \frac{\dfrac{9x^2}{5 \cdot 7}}{1} + \cdots,$$

to which Theorem 3.2 is applicable with $s_n = n/(2n+1)$. We find that the modulus of the function does not exceed $1 + |x^2|$ if $|x| \leqq 1$.

In the proof of the next theorem we use for the first time the *second* interpretation of the fundamental inequalities, which was mentioned in the introduction.

THEOREM C. *The continued fraction of Theorem A converges at least in the wider sense if*

$$(3.4) \qquad\qquad \liminf_{n = \infty} (r_1 r_2 r_3 \cdots r_n) = 0,$$

and converges to a finite value if (3.4) holds and actual inequality holds at least once in (1.3).

Proof. We need only consider the case where $a_n \neq 0$, $n = 2, 3, 4, \cdots$, since Theorem A disposes of the case where some a_n is 0. If $a_n \neq 0$ for all n, it is clear that $r_n > 0$ for all n.

Consider the continued fractions

$$(3.5) \qquad \frac{1}{1 + a_2} - \frac{a_2 a_3}{1 + a_3 + a_4} - \frac{a_4 a_5}{1 + a_5 + a_6} - \cdots,$$

$$(3.6) \qquad 1 - \frac{a_2}{1 + a_2 + a_3} - \frac{a_3 a_4}{1 + a_4 + a_5} - \frac{a_5 a_6}{1 + a_6 + a_7} - \cdots.$$

* Perron, p. 351, formula (17) (in slightly different form).

The approximants of (3.5) are the *even* approximants[*] of (1.1), while those of (3.6) are the *odd* approximants of (1.1). We now write the inequalities (1.3) in the form

(3.7)
$$\frac{r_1}{|a_2|}\left|1 + a_2\right| \geq 1,$$

$$\frac{r_{2n+1}}{|a_{2n+2}|}\left|1 + a_{2n+1} + a_{2n+2}\right| \geq \frac{r_{2n+1}}{|a_{2n+2}|}\frac{r_{2n-1}}{|a_{2n}|}\left|a_{2n}a_{2n+1}\right| + 1;$$

(3.8)
$$\frac{r_2}{|a_3|}\left|1 + a_2 + a_3\right| \geq 1,$$

$$\frac{r_{2n+2}}{|a_{2n+3}|}\left|1 + a_{2n+2} + a_{2n+3}\right| \geq \frac{r_{2n+2}}{|a_{2n+3}|}\frac{r_{2n}}{|a_{2n+1}|}\left|a_{2n+1}a_{2n+2}\right| + 1,$$

$n = 1, 2, 3, \cdots$. *But the conditions* (3.7) *constitute a Pringsheim test for* (3.5) *provided inequality holds at least once*;[†] and the same is true for conditions (3.8) and the continued fraction (3.6). It should be noted that the continued fractions (3.5), (3.6) may diverge to ∞ if at every stage in the conditions (3.7), (3.8) equality holds. In any case the limits $\lim_{n=\infty} (A_{2n}/B_{2n}) = L_0$, $\lim_{n=\infty} (A_{2n-1}/B_{2n-1}) = L_1$ exist, finite or infinite.

But by Theorem A,

$$\left|\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}\right| \leq r_1 r_2 \cdots r_n$$

so that by (3.4), $L_0 = L_1$. If inequality holds at least once in (1.3), then this common limit of the two sequences is finite, and the continued fraction converges.

In a recent paper, Leighton[‡] obtained a convergence theorem for (1.1) by applying the Pringsheim inequalities to (3.7) and (3.8) separately. Since the element a_1 appearing in his work can have no bearing upon the convergence of the continued fraction, it will be seen that his result may be stated as follows:

THEOREM 3.3. (Leighton.) *Let*[§]

(3.9)
$$\left|1 + a_2\right| \geq 1, \qquad \left|1 + a_2 + a_3\right| \geq 1,$$
$$\left|1 + a_n + a_{n+1}\right| \geq \left|a_{n-1}a_n\right| + 1,$$

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[*] Perron, p. 201, formulas (7), (8).

[†] See the second footnote.

[‡] Walter Leighton, Duke Mathematical Journal, vol. 4 (1938), pp. 775–778.

[§] The proof as given by Leighton would not permit equality in the first two relations, which, however, we allow only when some a_n vanishes.

$n = 3, 4, 5, \cdots$. *Then the continued fraction* (1.1) *converges if some a_n vanishes, or if actual inequality holds in the first two relations* (3.9) *and* $\lim \inf |a_n| < \infty$.

Proof. Multiply the inequalities (3.9) by $|a_2|$, $|a_3|$, $|a_4|$, $\cdots$, respectively, and they take the form (1.3) where $r_n = |a_{n+1}|$. Hence if some a_n vanishes, the continued fraction converges by Theorem A.

If $a_n \neq 0$, $n \geq 2$, then by hypothesis actual inequality holds in the first two relations (1.3), so that (3.5), (3.6) converge. Moreover, the quantities

$$g_1 = \Big| B_2 \Big| - \frac{|a_2|}{r_1} |B_0|, \qquad g_2 = \Big| B_3 \Big| - \frac{|a_3|}{r_2} |B_1|$$

are *positive*. We then have by (2.1), (1.3), the inequalities

$$\left\{ |B_{k+2}| - \frac{|a_{k+2}|}{r_{k+1}} |B_k| \right\} \geq r_{k-1} |a_{k+1}| \left\{ |B_k| - \frac{|a_k|}{r_{k-1}} |B_{k-2}| \right\}, \qquad k \geq 2,$$

and therefore

(3.10)
$$\left| B_{2n+2} \right| - \frac{|a_{2n+2}|}{r_{2n+1}} |B_{2n}| \geq r_1 r_3 \cdots r_{2n-1} |a_3 a_5 \cdots a_{2n+1}| g_1 > 0,$$

$$\left| B_{2n+3} \right| - \frac{|a_{2n+3}|}{r_{2n+2}} |B_{2n+1}| \geq r_2 r_4 \cdots r_{2n} |a_4 a_6 \cdots a_{2n+2}| g_2 > 0.$$

When $r_n = |a_{n+1}|$, these inequalities become

(3.11)
$$|B_{2n+2}| - |B_{2n}| \geq |a_2 a_3 \cdots a_{2n+1}| g_1,$$

$$|B_{2n+3}| - |B_{2n+1}| \geq |a_3 a_4 \cdots a_{2n+2}| g_2.$$

Consequently,

(3.12)
$$\left| \frac{A_{2n+2}}{B_{2n+2}} - \frac{A_{2n+1}}{B_{2n+1}} \right| \leq \frac{|a_2|}{g_2} \frac{1}{|B_{2n+2}|} \left(\left| \frac{B_{2n+3}}{B_{2n+1}} \right| - 1 \right),$$

$$\left| \frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}} \right| \leq \frac{1}{g_1} \frac{1}{|B_{2n+1}|} \left(\left| \frac{B_{2n+2}}{B_{2n}} \right| - 1 \right).$$

Since by (3.11) the sequences $|B_{2n}|$ and $|B_{2n-1}|$ are strictly monotone increasing, they must have limits, finite or infinite. If either has a finite limit, we see from (3.12), remembering that (3.5), (3.6) both converge, that the continued fraction (1.1) converges. If $\lim |B_{2n}| = \lim |B_{2n-1}| = \infty$, we have the inequalities

$$\left| \frac{B_{2n+2}}{B_{2n} B_{2n+1}} \right| \leq \frac{1}{|B_{2n}|} + \frac{|a_{2n+2}|}{|B_{2n+1}|}, \qquad \left| \frac{B_{2n+3}}{B_{2n+1} B_{2n+2}} \right| \leq \frac{1}{|B_{2n+1}|} + \frac{|a_{2n+3}|}{|B_{2n+2}|},$$

so that under the hypothesis lim inf $|a_n| < \infty$, the right-hand member of at least one of (3.12) must have the inferior limit 0, and, therefore, the continued fraction converges in this case also.

Of course, the inequalities (3.10) hold whenever the $r_n > 0$ exist satisfying (1.3) with inequality for $n = 1$, 2. We therefore have

$$\left| \frac{A_{2n+3}}{B_{2n+3}} - \frac{A_{2n+2}}{B_{2n+2}} \right| = \frac{|a_2 a_3 \cdots a_{2n+3}|}{|B_{2n+2} B_{2n+3}|}$$

$$\leqq \frac{|a_{2n+3}|}{g_1 g_2 r_1 r_2 \cdots r_{2n}} \left\{ 1 - \frac{|a_{2n+2}|}{r_{2n+1}} \left| \frac{B_{2n}}{B_{2n+2}} \right| \right\} \left\{ 1 - \frac{|a_{2n+3}|}{r_{2n+2}} \left| \frac{B_{2n+1}}{B_{2n+3}} \right| \right\}$$

$$< \frac{|a_{2n+3}|}{g_1 g_2 r_1 r_2 \cdots r_{2n}},$$

and, similarly,

$$\left| \frac{A_{2n+2}}{B_{2n+2}} - \frac{A_{2n+1}}{B_{2n+1}} \right| < \frac{|a_{2n+2}|}{g_1 g_2 r_1 r_2 \cdots r_{2n-1}}.$$

Now if lim inf $(r_1 r_2 \cdots r_n) = 0$, (1.1) converges by Theorem C. If $r_1 r_2 \cdots r_n$ is bounded away from 0, these inequalities show that if lim inf $|a_n| = 0$ the continued fraction converges. Hence we have proved

THEOREM 3.4. *If in the continued fraction* (1.1) *the inequalities* (1.3) *hold with inequality for $n = 1$, 2, then the continued fraction converges if* lim inf $|a_n| = 0$.

4. Convergence theorems involving $\sum |b_n|$. If the elements a_n are different from 0, the continued fraction (1.1) can be thrown into the form $1/b_1 + 1/b_2 + 1/b_3 + \cdots$, where $b_1 = 1$, $a_n = 1/b_n b_{n-1}$, $n = 2, 3, 4, \cdots$. Therefore, by a theorem of von Koch,[*] *the even and odd approximants of the continued fraction will have distinct limits, finite or infinite, if the series $\sum |b_n|$ converges. Then a necessary condition for the convergence of the continued fraction is the divergence of the series $\sum |b_n|$.*

In this section we shall give some conditions under which the divergence of the series $\sum |b_n|$ is *sufficient* to insure the convergence of the continued fraction.

THEOREM D. *If there exist positive numbers r_n satisfying the inequalities* (1.3), *with actual inequality holding for at least one even and one odd index, and if*

(4.1) $r_1 r_3 r_5 \cdots r_{2n-1} < M, \qquad r_2 r_4 r_6 \cdots r_{2n} < M,$

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* Perron, p. 235, Theorem 6.

$n = 1, 2, 3, \cdots$, *where M is a finite constant, then a necessary and sufficient condition for the convergence of the continued fraction*

(4.2) $1/1 + a_2/1 + a_3/1 + a_4/1 + \cdots$, $a_n \neq 0$,

is that the series $\sum |b_n|$ diverge, where $b_1 = 1$, $a_n = 1/b_{n-1}b_n$, $n = 2, 3, 4, \cdots$.

Proof. We may assume that actual inequality holds in (1.3) for $n = 1, 2$. For if inequality holds for a certain index n, we may slightly increase $r_{n-2}, r_{n-4}, \cdots$ in such a way as to introduce inequality for $n = 1, 2$. This cannot affect condition (4.1) since the values of only a finite number of the r_n have been changed.

In terms of the b_n the continued fraction (4.2) takes the form

(4.3) $1/b_1 + 1/b_2 + 1/b_3 + \cdots$.

If P_n/Q_n is the nth approximant of (4.3) one may easily verify the relation

(4.4) $b_{n+1}Q_{n+3} = (b_{n+1}b_{n+2}b_{n+3} + b_{n+1} + b_{n+3})Q_{n+1} - b_{n+3}Q_{n-1}$,

$n = 1, 2, 3, \cdots$. From this we obtain by means of (1.3) the inequality

(4.5) $|Q_{n+3}| - \dfrac{1}{r_{n+2}}|Q_{n+1}| \geqq r_n \left|\dfrac{b_{n+3}}{b_{n+1}}\right| \left(|Q_{n+1}| - \dfrac{1}{r_n}|Q_{n-1}|\right)$,

$n = 1, 2, 3, \cdots$. Since (1.3) holds with actual inequality for $n = 1, 2$, the quantities

$$e_1 = \frac{|Q_2| r_1 - |Q_0|}{|b_2| r_1}, \qquad e_2 = \frac{|Q_3| r_2 - |Q_1|}{|b_3| r_2}$$

are positive. On introducing the proper factors in (4.5) and summing we get

(4.6) $r_3 r_5 \cdots r_{2n+1} |Q_{2n+2}| \geqq e_1 \left(1 + \displaystyle\sum_{k=1}^{n} r_1 r_3^2 r_5^2 \cdots r_{2k-1}^2 r_{2k+1} |b_{2k+2}|\right)$

and

(4.7) $r_4 r_6 \cdots r_{2n+2} |Q_{2n+3}| \geqq e_2 \left(1 + \displaystyle\sum_{k=1}^{n+1} r_2 r_4^2 r_6^2 \cdots r_{2k-2}^2 r_{2k} |b_{2k+1}|\right)$.

Now

(4.8) $\left|\dfrac{P_n}{Q_n} - \dfrac{P_{n+1}}{Q_{n+1}}\right| = \dfrac{1}{|Q_n Q_{n+1}|}$;

and the limits $\lim_{n=\infty} (P_{2n}/Q_{2n})$, $\lim_{n=\infty} (P_{2n-1}/Q_{2n-1})$ both exist and are finite. *Hence the continued fraction will converge if $1/|Q_n Q_{n+1}|$ has the lower limit 0.* By Theorem C, if $r_1 r_2 \cdots r_n$ has the lower limit 0, our theorem is

granted. Hence there remains only the case where, for a constant $c > 0$,

$$c \leqq r_3 r_5 \cdots r_{2n-1} < M, \qquad c \leqq r_4 r_6 \cdots r_{2n} < M.$$

The theorem now follows from a consideration of (4.6), (4.7), and the hypothesis that the series $\sum |b_n|$ is divergent.

THEOREM E. *If the numbers a_n in (1.1) are all less in absolute value than a constant M, and if there exist positive numbers r_n satisfying the inequalities (1.3) with actual inequality for an even and an odd index, then the continued fraction converges.*

Proof. We may suppose the $a_n \neq 0$, for this case is treated in Theorem A. Furthermore we may suppose that $\liminf (r_1 r_2 \cdots r_n) \geqq c > 0$, for otherwise Theorem C applies. From a composition of corresponding terms in the inequalities (4.6), (4.7) we get

$$r_3 r_4 \cdots r_{n+1} | Q_{n+1} Q_{n+2} | \geqq e_1 e_2 (1 + \sum r_1 r_2 r_3^2 r_4^2 \cdots r_{n-1}^2 r_n r_{n+1} | b_{n+1} b_{n+2} |).$$

If $\limsup (r_1 r_2 \cdots r_n) \leqq N$, where N is finite, then

$$\limsup | Q_{n+1} Q_{n+2} | \geqq \frac{e_1 e_2}{N} \left(r_1 r_2 + \sum_{k=1}^{\infty} r_1^2 r_2^2 \cdots r_{k-1}^2 r_k r_{k+1} M \right),$$

and it is evident that the series on the right diverges. Then we have $\limsup | Q_{n+1} Q_{n+2} | = \infty$. If N is infinite we have $| Q_{n+1} Q_{n+2} | > e_1 e_2 (r_1 r_2 \cdots r_{n+1}) M$, and $\limsup | Q_{n+1} Q_{n+2} | = \infty$. The theorem now follows from (4.8).

5. **Convergence regions.** By a *convergence region* for the continued fraction (1.1) we shall understand a set W of points in the complex plane such that if $a_2, a_3, a_4, \cdots$ are arbitrary points in W the continued fraction converges. A convergence region is necessarily *bounded*. For if W were unbounded, we could choose the $a_n \neq 0$ in W in such a way that the series $\sum |b_n|$ would be convergent, where $b_1 = 1$, $a_n = 1/b_{n-1} b_n$, and for this choice of the a_n the continued fraction would diverge by oscillation. On the other hand, if the $a_n \neq 0$ lie in any bounded region, it is clear that the series $\sum |b_n|$ must *diverge*.

Worpitzky* showed that the interior and boundary of the circle $|z| = 1/4$ is a convergence region for (1.1); and since (1.1) diverges if $a_n = -1/4 - c$ where c is real and positive, it follows that the interior and boundary of a circle with center at the origin and radius greater than 1/4 cannot be a convergence region. Szász† obtained convergence regions containing the origin, which neither contain nor are contained in the circle $|z| = 1/4$.

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* See the second footnote on page 156.

† Szász, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 132–160.

We now propose to characterize completely all convergence regions which are symmetric with respect to the real axis.

In order to obtain a *necessary restriction* upon such a region, consider the continued fraction $1/1+\bar{z}/1+z/1+\bar{z}/1+z/1+\cdots$, in which $z=x+iy$, $\bar{z}=x-iy$, x, y real. If this continued fraction converges, then the continued fraction

$$(5.1) \qquad -z + \frac{z}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \cdots$$

must converge, as well as the continued fraction

$$\frac{|z^2|}{1+2x} - \frac{|z^2|}{1+2x} - \frac{|z^2|}{1+2x} - \cdots,$$

having as approximants the odd approximants of (5.1). Now if the latter converges, it must have a real value, which is a root of the quadratic equation in u:

$$u^2 - (1+2x)u + (x^2 + y^2) = 0.$$

Consequently, it is *necessary* for the convergence of (5.1) that $y^2 \leq x+1/4$, that is, z shall lie within or upon the parabola $|z| - \Re(z) = 1/2$, where $\Re(z)$ denotes the real part of z. *Therefore a convergence region for* (1.1) *which is symmetric with respect to the real axis is necessarily a finite region bounded by this parabola.* We shall prove that this necessary condition is also a sufficient condition.

THEOREM F. (The parabola theorem.) *If the elements a_n of the continued fraction* (1.1) *lie within or upon the parabola (see figure)*

$$(5.2) \qquad\qquad |z| - \Re(z) = 1/2,$$

then:

(a) *The denominator of the nth approximant A_n/B_n is different from 0 for all n.*

(b) *The sequences of even and odd approximants have finite limits L_0 and L_1, and $|L_0 - L_1| \leq 1$.*

(c) *If $a_n \neq 0$, $n = 2, 3, 4, \cdots$, the continued fraction converges if and only if the series $\sum |b_n|$ diverges, where $b_1 = 1$, $a_n = 1/b_n b_{n-1}$, $n = 2, 3, 4, \cdots$.*

(d) *If some a_n vanishes, the continued fraction converges and equals one of its approximants.*

(e) *The parabola* (5.2) *is the best possible curve symmetric with respect to the real axis having these properties.*

Proof. Part (e) follows from the fact shown above that when $a_{2n} = \bar{z}$, $a_{2n-1} = z$, the continued fraction diverges if z lies outside the parabola.

The other parts of the theorem will follow from Theorems A and D, if we show that the inequalities (1.3) hold for $r_n = 1$, with actual inequality for $n = 1, 2$.
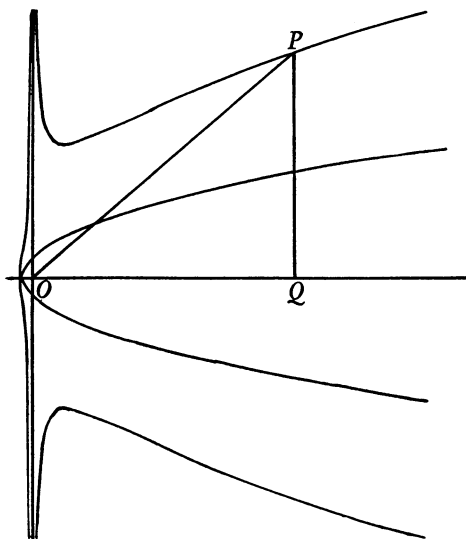
Put $a_n = u_n + i v_n$ where u_n, v_n are real. Then if a_n lies in or upon the parabola (5.2), we must have

$$| a_n | = u_n + h_n/2, \qquad n = 2, 3, 4, \cdots ,$$

where $0 \leq h_n \leq 1$. We then have

$$| 1 + a_2 | \geq 1 + u_2 > u_2 + h_2/2 = | a_2 | ,$$

$$| 1 + a_2 + a_3 | \geq 1 + u_2 + u_3 = (1 - h_2/2 - h_3/2) + | a_2 | + | a_3 | > | a_3 | ,$$

$$| 1 + a_n + a_{n+1} | \geq 1 + u_n + u_{n+1} \geq u_n + h_n/2 + u_{n+1} + h_{n+1}/2$$

$$\geq | a_n | + | a_{n+1} | ,$$

$n = 3, 4, 5, \cdots$, as was to be proved.

It is of interest to note the analogy between the above result and the following important and general convergence criterion of Van Vleck:[*]

THEOREM 5.1. (Van Vleck.) *Let* $b_n = | b_n | e^{i\theta_n}$ *where*

$$- \pi/2 + \epsilon \leq \theta_n \leq \pi/2 - \epsilon, \qquad \epsilon > 0,$$

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* Perron, pp. 264–271.

$n = 1, 2, 3, \cdots$, *and suppose not all the* b_n *with odd index are* 0. *The then sequences of even and odd approximants of the continued fraction*

$$1/b_1 + 1/b_2 + 1/b_3 + \cdots$$

have finite limits; and the continued fraction converges if and only if the series $\sum |b_n|$ *diverges.*

It is interesting to note that the expression $|a_n| - \Re(a_n)$ appearing in the parabola theorem occurs in the following theorem of Szász:*

THEOREM 5.2. (Szász.) *The continued fraction* (1.1) *converges if the series* $\sum |a_n|$ *converges, and*

$$\sum_2^\infty \{ |a_n| - \Re(a_n) \} < 2.$$

It is easy to construct examples where convergence can be established by Theorem 5.1 but not by Theorem A or our later theorems, and vice versa. The same statement applies to Theorem 5.2, where, however, at most *three* of the a_n's can lie outside the parabola. But if, for example, a_2 is taken outside the parabola equal to $-1+\epsilon$, and $a_3 = a_4 = 1$, then if the other a_n's are chosen so that

$$\sum_5^\infty |a_n| - \sum_5^\infty \Re(a_n) < 2\epsilon, \qquad\qquad \epsilon > 0,$$

the conditions of Szász are satisfied, but the inequalities (1.3) cannot hold for nonnegative r_n. In fact, we must have $r_1 \geq |a_2| / |1+a_2| = 1/\epsilon - 1$. But $|1+a_3+a_4| < r_1 |a_3|$ if $\epsilon < 1/4$, and hence it is not possible to determine $r_3 \geq 0$.

We have seen that a convergence region for (1.1) which is symmetric with respect to the real axis must lie in the parabola (5.2). We now turn our attention to the problem of finding convergence regions which are not symmetric with respect to the real axis.

THEOREM G. (The triangle theorem.) *Let* $P(x, y)$ *be any point upon the curve* (*see figure after Theorem* F)

$$(5.3) \qquad\qquad y = (2x + 1)(4x + 1)^{1/2}/2x,$$

in the plane of $z = x + iy$. *Let* O *denote the origin and* Q *the point* $z = x$. *Then the closed region made up of the interior and boundary of the triangle* OPQ *is a convergence region for the continued fraction* (1.1).

Proof. As in the proof of Theorem F, we show that Theorem D applies

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with $r_n = 1$. Let m be the slope of the line OP. Then if a_n is in the triangle OPQ, we may write $a_n = u_n(1 + imt_n)$, where u_n is the real part of a_n, and $0 \leq t_n \leq 1$. Since $u_2 > -1/2$ it follows that $|1 + a_2| > |a_2|$, so that the first inequality (1.3) holds with $r_1 = 1$. Since $u_n \geq -1/4$, $u_2u_3 \geq 0$, $v_2v_3 \geq 0$, where $v_n i$ is the imaginary part of a_n, it is easily seen that $|1 + a_2 + a_3| > |a_3|$, so that the second inequality (1.3) holds. Thus actual inequality holds in the first two relations (1.3). The remaining inequalities will hold provided

$$(5.4) \quad \begin{aligned} (1 + 2u_n + 2u_{n+1} + 2u_n u_{n+1}) \\ \geq 2u_n u_{n+1}[(1 + m^2 s^2)^{1/2}(1 + m^2 t^2)^{1/2} - 2m^2 s^2 t^2], \end{aligned}$$

$n = 3, 4, 5, \cdots$, where $s = t_n$, $t = t_{n+1}$. We find that the maximum of the bracketed part in the right-hand member of (5.4) for s, t between 0 and 1 is $(1 + m^2)^{1/2}$. Hence (5.4) will hold if

$$(5.5) \quad [(1 + m^2)^{1/2} - 1]u_n u_{n+1} - u_n - u_{n+1} \leq 1/2.$$

We next maximize the left-hand member of (5.5) in the two cases $u_n > 0$, $u_n < 0$, and find that in the first case (5.5) will hold if $w = u_n$ satisfies the inequality

$$(5.6) \quad w \leq [2\{(1 + [1 + m^2]^{1/2})\}^{1/2} - 2]^{-1},$$

while in the second case (5.5) will hold provided $w = -u_n$ satisfies (5.6). But this is precisely the condition that the point P shall lie upon the curve (5.3). The proof of Theorem G is now complete.

It is easy to show that *any bounded set of points lying upon a ray from the origin, in the right half-plane, is a convergence region for* (1.1). In fact, upon such a ray the inequalities (1.3) hold for $r_n = 1$ with actual inequality throughout. To see this, put $a_n = \alpha_n e^{2i\theta_n}$ in (1.3), and the latter take the form

$$(5.7) \quad \begin{aligned} (1 + \alpha_n + \alpha_{n+1})^2 - \left(r_{n-2}\alpha_n + \frac{\alpha_{n+1}}{r_n} \right)^2 \\ \geq 4\{ \alpha_n \sin^2 \theta_n + \alpha_n \alpha_{n+1} \sin^2 (\theta_{n+1} - \theta_n) + \alpha_{n+1} \sin^2 \theta_{n+1} \}, \end{aligned}$$

$n = 1, 2, 3, \cdots$, $(r_0 = r_{-1} = \alpha_1 = 0)$. When $r_n = 1$, $\theta_n = \theta = \text{const.}$, these reduce to

$$1 + 2\alpha_2 \geq 0, \quad (\alpha_2 + \alpha_3) \cos 2\theta \geq - (1 + \alpha_2^2)/(\alpha_2 + \alpha_3),$$

$$(\alpha_n + \alpha_{n+1}) \cos 2\theta \geq - 1/2,$$

$n = 3, 4, 5, \cdots$. These are obviously satisfied with actual inequality throughout, if $-\pi/2 \leq 2\theta \leq \pi/2$.

As an application, consider an arbitrary sequence $x_1, x_2, x_3, \cdots$ with nonnegative real parts. Let v_n be the ray from the origin upon which x_n lies.

Choose y_n upon v_n but in the parabola (5.2). Then if $a_2 = x_1$, $a_3 = y_1$, $a_4 = y_2$, $a_5 = x_2$, $a_6 = y_2$, $a_7 = y_3$, $a_8 = x_3$, $\cdots$, it is clear that the inequalities (1.3) are satisfied with $r_n = 1$. If, in particular, the y_n have a limit point in the finite plane (and they can evidently be so chosen), then the continued fraction $1/1 + a_2/1 + a_3/1 + \cdots$ converges by Theorem D. *This shows that it is possible for the a_n to be everywhere dense in the right half-plane for a convergent continued fraction* (1.1).*

6. **The problem of Szász.**† Szász proposed the following general problem: *To assign neighborhoods to the elements of a given convergent continued fraction in which these elements may vary without destroying the convergence property of the continued fraction.* We shall indicate in this section how Theorem A may be applied to this problem.

We consider a convergent continued fraction

$$(6.1) \qquad \frac{1}{1} + \frac{c_2}{1} + \frac{c_3}{1} + \cdots$$

and seek to determine positive numbers $t_2, t_3, t_4, \cdots$ such that if

$$(6.2) \qquad |a_n - c_n| \leqq t_n, \qquad n = 2, 3, 4, \cdots,$$

then the continued fraction

$$\frac{1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots$$

will converge. The theorems of the preceding sections may be applied to this problem in certain cases. For that purpose put

$$a_n = c_n + \delta_n t_n, \qquad n = 2, 3, 4, \cdots, 0 \leqq |\delta_n| \leqq 1,$$

in the inequalities (1.3). Then these inequalities will hold provided

$$r_1 |1 + c_2| \geqq |c_2| + (1 + r_1)t_2,$$
$$(6.3) \quad r_2 |1 + c_2 + c_3| \geqq |c_3| + (1 + r_2)t_3 + r_2 t_2,$$
$$r_n |1 + c_n + c_{n+1}| \geqq r_n r_{n-2} |c_n| + |c_{n+1}| + (1 + r_n)t_{n+1} + r_n(1 + r_{n-2})t_n,$$

$n = 3, 4, 5, \cdots$. It follows that if there exist positive numbers r_n, t_n satisfying (6.3), then

$$(6.4) \qquad r_n |1 + c_n + c_{n+1}| \geqq r_n r_{n-2} |c_n| + |c_{n+1}|,$$

$n = 1, 2, 3, \cdots$, $(r_{-1} = r_0 = a_1 = 0)$, and hence under suitable restrictions, (1.1)

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* Leighton and Wall showed that this can be done for the entire plane. See American Journal of Mathematics, vol. 58 (1936), pp. 267–281; p. 269.

† See the second footnote on page 165.

will converge if (6.2) holds. If the a_n, c_n, t_n are functions of variables, and the r_n are independent of the variables, then under suitable conditions the convergence will be *uniform*.

It is convenient to specialize the inequalities (6.3) in the following way. Put

$$(6.5) \qquad s_1 = \frac{r_1 \big| 1 + c_2 \big| - \big| c_2 \big|}{1 + r_1}, \qquad s_2 = \frac{r_2 \big| 1 + c_2 + c_3 \big| - \big| c_3 \big|}{1 + 2r_2}$$

$$s_n = \frac{r_n \big| 1 + c_n + c_{n+1} \big| - r_n r_{n-2} \big| c_n \big| - \big| c_{n+1} \big|}{1 + 2r_n + r_n r_{n-2}},$$

$n = 3, 4, 5, \cdots$. If $s_1, s_2, s_3, \cdots \geqq 0$, and t_n' is the larger of s_{n-1}, s_n, then $t_n = t_n'$, $(n \geqq 2)$, satisfy the inequalities (6.3). We therefore have this theorem:

THEOREM 6.1. *Let* $a_2, a_3, a_4, \cdots, c_2, c_3, c_4, \cdots$ *be functions of any variables over a domain* D, *and suppose that there exist numbers* $r_1, r_2, r_3, \cdots$, *nonnegative and independent of the variables such that the quantities* s_n, *defined by* (6.5), *are nonnegative over* D. *Then if*

$$(6.6) \qquad \big| a_n - c_n \big| \leqq \begin{cases} s_{n-1}, \\ s_n \end{cases} \qquad n = 2, 3, 4, \cdots,$$

we shall have $r_n \big| 1 + a_n + a_{n+1} \big| \geqq r_n r_{n-2} \big| a_n \big| + \big| a_{n+1} \big|$ *for* $n = 1, 2, 3, \cdots$, $(r_{-1} = r_0 = a_1 = 0)$. *Hence, in particular, if* $1 + \sum r_1 r_2 \cdots r_n$ *converges, the continued fraction* (1.1) *will converge uniformly over* D.

As an illustration, let $c_n = (2n - 3)/4$, $n \geqq 2$; $r_n = n/(n+2)$, $n \geqq 1$. Then $s_1 = 1/8$, $s_n = (n-1)/(4n)$, $n \geqq 2$, and hence the continued fraction (1.1) in which $a_2 = (1 + 4x_2)/4$, $a_3 = (3 + 4x_3)/4$, $\cdots$, $a_n = (2n - 3 + 4x_n)/4$, $\cdots$ converges uniformly for

$$\big| x_2 \big| \leqq 1/8, \qquad \big| x_n \big| \leqq 1/4 - 1/[4(n-1)],$$

$n = 3, 4, 5, \cdots$.

If $c_n = c$ and $\Re(c) > -1/4$, we can determine by means of Theorem 6.1 a neighborhood of c which is a convergence region for (1.1).

THEOREM 6.2. *Let* $c, a_2, a_3, a_4, \cdots$ *be functions of any variables over a certain domain* D *in which*

$$\big| a_n - c \big| \leqq \frac{\big| 1 + 2c \big| - 2 \big| c \big|}{4}, \qquad n = 2, 3, 4, \cdots.$$

Then the continued fraction (1.1) *converges uniformly over* D.

Proof. Take $r_n = 1$, $c_n = c$ in Theorem 6.1. Then the theorem follows at once by application of Theorem D.

It is not difficult to show by means of Theorem A, with $r_n = n/(n+2)$, that 2 is the least upper bound of the absolute value of the continued fraction in Theorem 6.2.

Theorem 6.2 gives a generalization of the Worpitzky theorem, which reduces to the latter when $c = 0$. It is applicable whenever the real part of c exceeds $-1/4$, and in this case furnishes a solution of the problem of Szász for the periodic continued fraction $1/1 + c/1 + c/1 + \cdots$. Szász obtained a solution except when c is real and not greater than $-1/4$, in which case there is no solution. We note that when $c = 1$, the theorem of Szász gives as the radius of the convergence circle approximately $5/22$, Theorem 6.1 gives the value $1/4$, and the *parabola theorem* gives the value 1. Moreover, from the latter theorem it follows that the radius of a convergence circle with center $c = 1$ cannot exceed 1.

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